P216 Mathematical Physics
Quiz 1, October 19 2017, Time: 90 minutes

Solve 8 problems out of 10

1. If $\overrightarrow{A}$ and $\overrightarrow{B}$ are constant vectors, show that $\overrightarrow{\nabla}\left(\overrightarrow{A}\cdot\left(\overrightarrow{B}\times\overrightarrow{r}\right)\right)=\overrightarrow{A}\times\overrightarrow{B}$. Hint: cyclicity of triple products.

2. Consider the vector $\overrightarrow{E}=r^{n-1}\overrightarrow{r}$. Verify Gauss' theorem for the volume and surface of a sphere of radius R.

3. Consider the vector $\overrightarrow{A}=\rho\widehat{\varphi}+z\widehat{z}$ where $\widehat{\varphi}$ and $\widehat{z}$ are unit vectors in cylindrical coordinates. Verify Stokes theorem for the surface of a disk of radius R centered at $z=0$ bounded by the circle of radius R and with the same center.

4. Evaluate the integral $\int\limits_{-\infty}^{\infty} f(x)\,\delta\left(x^2-x-6\right)dx$. Warning: Delta function is even $\delta(-x)=\delta(x)=\delta(|x|)$

5. Find eigenvalues and normalized eigenvectors of the matrix

$$A=\begin{pmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{pmatrix}$$

   Express the eigenvectors in terms of $\sin\frac{\theta}{2}$ and $\cos\frac{\theta}{2}$.

6. Prove that $\int\limits_S d\overrightarrow{a}\times\overrightarrow{\nabla}\phi=\oint\limits_C \phi d\overrightarrow{l}$ where $C=\partial S$. Hint: Take scalar product with arbitrary constant vector $\overrightarrow{c}$.

7. Let A be a linear operator mapping the complex vector V_3 into itself with the matrix representation $A=\begin{pmatrix}1 & 2 & -i\\ 2 & 1 & i\\ i & -i & 1\end{pmatrix}$ and $\{\overrightarrow{e}_i\}$, $i=1,2,3$ be a set of three basis vectors. This basis is mapped into a new set $\{\overrightarrow{e}'_i\}$ where $\overrightarrow{e}'_i=\sum\limits_{j=1}^{3}A_{ji}\overrightarrow{e}_j$ where A_{ij} are the matrix elements of A. Write explicitly the new basis elements $\overrightarrow{e}'_1$, $\overrightarrow{e}'_2$, $\overrightarrow{e}'_3$ in terms of $\overrightarrow{e}_1$, $\overrightarrow{e}_2$, $\overrightarrow{e}_3$. Does the set $\{\overrightarrow{e}'_i\}$ form a basis?

8. Find all inner products of the two vectors $\overrightarrow{x}=\begin{pmatrix}1+i\\ 1-i\\ 2\end{pmatrix}$, $\overrightarrow{y}=\begin{pmatrix}1-i\\ 1+i\\ -2\end{pmatrix}$.

9. Show that a necessary and sufficient condition for two linear operators A and B to have a simultaneous eigenvector $\overrightarrow{x}$ is that they must commute, i.e. $[A,B]=0$.

10. Find the determinant, matrix of cofactors of and inverse of the matrix A where

$$A=\begin{pmatrix}1 & 2 & -3\\ 2 & 5 & -4\\ -3 & -4 & 8\end{pmatrix}$$